

5.1 Duality

$$\forall x A(x) \leftrightarrow \neg \exists x \neg A(x) \quad \exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$$

5.2 Commutativity and Distributivity

$$\begin{aligned} \forall x \forall y A(x, y) &\leftrightarrow \forall y \forall x A(x, y) \quad \exists x \exists y A(x, y) \leftrightarrow \exists y \exists x A(x, y) \\ \exists x \forall y A(x, y) &\rightarrow \forall y \exists x A(x, y) \\ &\models \forall x A(x) \vee \forall x B(x) \rightarrow \forall x (A(x) \vee B(x)) \\ &\models \exists x (A(x) \wedge B(x)) \rightarrow \exists x A(x) \wedge \exists x B(x) \end{aligned}$$

Suppose $D = \{d_1, d_2\}$, and

$$\begin{aligned} v(A(d_1)) &= T, & v(A(d_2)) &= F, \\ v(B(d_1)) &= F, & v(B(d_2)) &= T. \end{aligned}$$

Then, $v(\exists x A(x) \wedge \exists x B(x)) = T$, but $v(\exists x (A(x) \wedge B(x))) = F$.

5.3 Quantification Without the Free Variable in its scope

$$\begin{aligned} (\exists x A(x) \vee B) &\leftrightarrow \exists x (A(x) \vee B) \quad (\forall x A(x) \vee B) \leftrightarrow \forall x (A(x) \vee B) \\ (B \vee \exists x A(x)) &\leftrightarrow \exists x (B \vee A(x)) \quad (B \vee \forall x A(x)) \leftrightarrow \forall x (B \vee A(x)) \\ (\exists x A(x) \wedge B) &\leftrightarrow \exists x (A(x) \wedge B) \quad (\forall x A(x) \wedge B) \leftrightarrow \forall x (A(x) \wedge B) \\ (B \wedge \exists x A(x)) &\leftrightarrow \exists x (B \wedge A(x)) \quad (B \wedge \forall x A(x)) \leftrightarrow \forall x (B \wedge A(x)) \end{aligned}$$

5.4 Quantification over Implication and Equivalence

$$\begin{aligned}\forall x(A \rightarrow B(x)) &\leftrightarrow (A \rightarrow \forall xB(x)) \\ \forall x(A(x) \rightarrow B) &\leftrightarrow (\exists xA(x) \rightarrow B)\end{aligned}$$

$$\begin{aligned}(\exists x(A(x) \vee B(x)) &\leftrightarrow (\exists xA(x) \vee \exists xB(x)) \\ \forall x(A(x) \wedge B(x)) &\leftrightarrow (\forall xA(x) \wedge \forall xB(x)) \\ \forall xA(x) \vee \forall xB(x) &\rightarrow \forall x(A(x) \vee B(x)) \\ \exists x(A(x) \wedge B(x)) &\rightarrow (\exists xA(x) \wedge \exists xB(x))\end{aligned}$$

$$\begin{aligned}\forall x(A(x) \leftrightarrow B(x)) &\rightarrow (\forall xA(x) \leftrightarrow \forall xB(x)) \\ \forall x(A(x) \leftrightarrow B(x)) &\rightarrow (\exists xA(x) \leftrightarrow \exists xB(x))\end{aligned}$$

$$\begin{aligned}\exists x(A(x) \rightarrow B(x)) &\leftrightarrow (\forall xA(x) \rightarrow \exists xB(x)) \\ (\exists xA(x) \rightarrow \forall xB(x)) &\rightarrow \forall x(A(x) \rightarrow B(x))\end{aligned}$$

$$\begin{aligned}\forall x(A(x) \vee B(x)) &\rightarrow (\forall xA(x) \vee \exists xB(x)) \\ \forall x(A(x) \rightarrow B(x)) &\rightarrow (\forall xA(x) \rightarrow \forall xB(x))\end{aligned}$$

$$\begin{aligned}\forall x(A(x) \rightarrow B(x)) &\rightarrow (\exists xA(x) \rightarrow \exists xB(x)) \\ \forall x(A(x) \rightarrow B(x)) &\rightarrow (\forall xA(x) \rightarrow \exists xB(x))\end{aligned}$$