推导与希尔伯特系统

一个**推导系统**(Deductive System)由一组称为**公理**(Axioms)的公式和一组**推导规则**(Rules of Inference)构成。

在推导系统中,一个**证明**(Proof)由一个公式序列构成:

$$S = \{A_1, \cdots, A_n\}$$

其中,每一个公式A_i要么是公理,要么是由前面公式$A_{j1}, \cdots, A_{jk}$ $(j_1 < \cdots < j_k < i)$根据规则推导出来的公式。

序列里的最后一个公式A_n称为**定理**(Theorem), 序列S称为A_n的一个**证明**(Proof). 称A_n是**可证明的**(Provable),表示成$\vdash A_n$.

如果$\vdash A$, 那么A可当作公理一样用于其它定理的证明。

希尔伯特系统是最早的推导系统之一。

希尔伯特系统($\mathcal{H}$)

公理 1: $\vdash (A \to (B \to A))$

公理 2: $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

公理 3: $\vdash (\neg B \to \neg A) \to (A \to B)$

假言推理(Modus Ponens): $\frac{\vdash A \qquad \vdash A \to B}{\vdash B}$.

由三个公理和一个推导规则构成。

Theorem $\vdash A \to A$

Proof.

1.	$\vdash (A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$	$A2$
2.	$\vdash A \to ((A \to A) \to A)$	$A1$
3.	$\vdash (A \to (A \to A)) \to (A \to A)$	$MP, 1,2$
4.	$\vdash A \to (A \to A)$	$A1$
5.	$\vdash A \to A$	$MP, 3,4$

在基本的$\mathcal{H}$系统中,也可以派生出其它规则。

如前提引入规则(Assumption and Deduction):

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \to B}$$

下面借助前提引入规则,证明定理:

$$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$$

证明：

1.	$\{A \to B, B \to C, A\} \vdash A$	$Assumption$
2.	$\{A \to B, B \to C, A\} \vdash A \to B$	$Assumption$
3.	$\{A \to B, B \to C, A\} \vdash B$	$MP, 1,2$
4.	$\{A \to B, B \to C, A\} \vdash B \to C$	$Assumption$
5.	$\{A \to B, B \to C, A\} \vdash C$	$MP, 3,4$
6.	$\{A \to B, B \to C\} \vdash A \to C$	$Deduction, 5$
7.	$\{A \to B\} \vdash [(B \to C) \to (A \to C)]$	$Deduction, 6$
8.	$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$	$Deduction, 7$

定理:$\vdash [A \to (B \to C)] \to [B \to (A \to C)]$.

证明:

1.	$A \to (B \to C), B, A \vdash A$	$Assumption$
2.	$A \to (B \to C), B, A \vdash A \to (B \to C)$	$Assumption$
3.	$A \to (B \to C), B, A \vdash B \to C$	$MP, 1,2$
4.	$A \to (B \to C), B, A \vdash B$	$Assumption$
5.	$A \to (B \to C), B, A \vdash C$	$MP, 3,4$
6.	$A \to (B \to C), B \vdash A \to C$	$Deduction, 5$
7.	$A \to (B \to C) \vdash B \to (A \to C)$	$Deduction, 6$
8.	$\vdash [A \to (B \to C)] \to [B \to (A \to C)]$	$Deduction, 7$

定理: $\vdash \neg A \to (A \to B)$.

证明:

1.	$\neg A \vdash \neg A \to (\neg B \to \neg A)$	$A1$
2.	$\neg A \vdash \neg A$	$Assumption$
3.	$\neg A \vdash \neg B \to \neg A$	$MP, 1,2$
4.	$\neg A \vdash (\neg B \to \neg A) \to (A \to B)$	$A3$
5.	$\neg A \vdash A \to B$	$MP, 3,4$
6.	$\vdash \neg A \to (A \to B)$	$Deduction, 5$

逆否规则(Contrapositive): $\frac{U\vdash\neg B\to\neg A}{U\vdash A\to B}$

定理: $\vdash\neg\neg A\to A$.

1.	$\neg\neg A\vdash\neg\neg A\to(\neg\neg\neg\neg A\to\neg\neg A)$	$A1$
2.	$\neg\neg A\vdash\neg\neg A$	$Assumption$
3.	$\neg\neg A\vdash\neg\neg\neg\neg A\to\neg\neg A$	$MP, 1,2$
4.	$\neg\neg A\vdash\neg A\to\neg\neg\neg A$	$Contrapositive, 3$
5.	$\neg\neg A\vdash\neg\neg A\to A$	$Contrapositive, 4$
6.	$\neg\neg A\vdash A$	$MP, 2,5$
7.	$\vdash\neg\neg A\to A$	$Deduction, 6$

定理: $\vdash A \to \neg\neg A$.

证明:

1. $\vdash \neg\neg\neg A \to \neg A$ $\quad$ *Proved*
2. $\vdash A \to \neg\neg A$ $\quad$ *Contrapositive*, 1

传递规则(Transitivity): $\frac{U \vdash A \to B \qquad U \vdash B \to C}{U \vdash A \to C}$

定理: $\vdash (A \to B) \to (\neg B \to \neg A)$.

证明:

1.	$A \to B \vdash A \to B$	$Assumption$
2.	$A \to B \vdash \neg\neg A \to A$	$Proved$
3.	$A \to B \vdash \neg\neg A \to B$	$Transitivity, 2,1$
4.	$A \to B \vdash B \to \neg\neg B$	$Proved$
5.	$A \to B \vdash \neg\neg A \to \neg\neg B$	$Transitivity, 3,4$
6.	$A \to B \vdash \neg B \to \neg A$	$Contrapositive, 5$
7.	$\vdash (A \to B) \to (\neg B \to \neg A)$	$Deduction, 6$

前提交换规则(Exchange of Antecedent):

$$\frac{U\vdash A\to(B\to C)}{U\vdash B\to(A\to C)}$$

定理:$\vdash A\to(\neg A\to B)$.

证明:

1. $\vdash \neg A\to(A\to B)$ $\quad$ $Proved$
2. $\vdash A\to(\neg A\to B)$ $\quad$ $Exchange, 1$

否定之否定规则(Double Negation):

$$\frac{U \vdash \neg\neg A}{U \vdash A}$$

反证规则(Reductio Ad Absurdum):

$$\frac{U \vdash \neg A \rightarrow false}{U \vdash A}$$

定理: $\vdash (A \to \neg A) \to \neg A$.

1.	$A \to \neg A, \neg\neg A \vdash \neg\neg A$	$Assumption$
2.	$A \to \neg A, \neg\neg A \vdash A$	$Double\ Negation, 1$
3.	$A \to \neg A, \neg\neg A \vdash A \to \neg A$	$Assumption$
4.	$A \to \neg A, \neg\neg A \vdash \neg A$	$MP, 2{,}3$
5.	$A \to \neg A, \neg\neg A \vdash A \to (\neg A \to false)$	$Proved$
6.	$A \to \neg A, \neg\neg A \vdash \neg A \to false$	$MP, 2{,}5$
7.	$A \to \neg A, \neg\neg A \vdash false$	$MP, 4{,}6$
8.	$A \to \neg A \vdash \neg\neg A \to false$	$Deduction, 7$
9.	$A \to \neg A \vdash \neg A$	$Reductio\ ad\ absurdum, 8$
10.	$\vdash (A \to \neg A) \to \neg A$	$Deduction, 9$

希尔伯特系统的变种:

Axiom 1

Axiom 2

Axiom 3′

$$Axiom\, 3' \quad \vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$$

1.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash \neg B$	$Assumption$
2.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash \neg B \to A$	$Assumption$
3.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash A$	$MP, 1,2$
4.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash \neg B \to \neg A$	$Assumption$
5.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash A \to B$	$Contrapositive, 4$
6.	$\neg B \to \neg A, \neg B \to A, \neg B \vdash B$	$MP, 3,5$
7.	$\neg B \to \neg A, \neg B \to A \vdash \neg B \to B$	$Deduction, 7$
8.	$\neg B \to \neg A, \neg B \to A \vdash (\neg B \to B) \to B$	$Proved$
9.	$\neg B \to \neg A, \neg B \to A \vdash B$	$MP, 8,9$
10.	$\neg B \to \neg A \vdash (\neg B \to A) \to B$	$Deduction, 9$
11.	$\vdash (\neg B \to \neg A) \to ((\neg B \to A) \to B)$	$Deduction, 10$

不一定是三个公理,以下是四个公理的推导系统:

$$Axiom1 \qquad \vdash A \vee A \to A$$

$$Axiom2 \qquad \vdash A \to A \vee B$$

$$Axiom3 \qquad \vdash A \vee B \to B \vee A$$

$$Axiom4 \qquad \vdash (B \to C) \to (A \vee B \to A \vee C)$$

梅瑞狄斯定理(Meredith’s Axiom):

$$\vdash (\{[(A \to B) \to (\neg C \to \neg D)] \to C\} \to E) \to [(E \to A) \to (D \to A)]$$