连分式

李 辉

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连分式

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

A (finite) **continued fraction**, where n is a non-negative integer, a_0 is an integer, and a_i is a positive integer, for $i = 1, \cdots, n$.

$$[a_0, a_1, \cdots, a_n]$$

Example

$$\frac{415}{93} = 4 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7}}}$$

$$[4,2,6,7]$$

Excise016

Try to convert $\frac{1999}{233}$ to form of continued fraction.

Relation to Euclid's Algorithm

$$415 = 4 \times 93 + 43$$
$$93 = 2 \times 43 + 7$$
$$43 = 6 \times 7 + 1$$
$$7 = 7 \times 1$$

$$[4,2,6,7]$$

Represention of Numbers

- Golden Ratio ϕ: $[1,1,1,\cdots]$

Proof:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\phi = \frac{1}{\phi - 1}$$

So,

$$\phi = 1 + (\phi - 1) = 1 + \frac{1}{\frac{1}{\phi - 1}} = 1 + \frac{1}{\phi}$$

- $\sqrt{2}$: $[1,2,2,2,\cdots]$

Proof:

$$1 + \sqrt{2} = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{\frac{1}{\sqrt{2} - 1}}$$

$$= 2 + \frac{1}{1 + \sqrt{2}} = [2,2,2,2,\cdots]$$

So,

$$\sqrt{2} = [1,2,2,2,\cdots]$$

Definition

$[a_0, a_1, \cdots, a_k]$ is **convergent** to $[a_0, a_1, \cdots, a_n]$, for $0 \le k \le n$.

Example

The first four convergents are

$$\left\{\frac{a_0}{1}, \frac{a_1 a_0 + 1}{a_1}, \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}, \frac{a_3(a_2(a_1 a_0 + 1) + a_0) + (a_1 a_0 + 1)}{a_3(a_2 a_1 + 1) + a_1}\right\}$$

$$\{p_0, p_1, p_2, p_3\} = \{a_0,\ a_1 a_0 + 1,\ a_2(a_1 a_0 + 1) + a_0,\ a_3(a_2(a_1 a_0 + 1) + a_0) + (a_1 a_0 + 1)\}$$

$$\{q_0, q_1, q_2, q_3\} = \{1, a_1, a_2 a_1 + 1, a_3(a_2 a_1 + 1) + a_1\}$$

$$p_k = a_k p_{k-1} + p_{k-2}$$
$$q_k = a_k q_{k-1} + q_{k-2}$$

^	0	1	2	3
a	4	2	6	7
p	4	9	58	415
q	1	2	13	93
p/q	4/1	9/2	58/13	415/93

^	-2	-1	0	1	2	3
a	^	^	4	2	6	7
p	0	1	4	9	58	415
q	1	0	1	2	13	93

Theorem $[a_0, a_1, \cdots, a_k] = \frac{p_k}{q_k}$

Proof.

$$\frac{p_0}{q_0} = \frac{a_0 p_{-1} + p_{-2}}{a_0 q_{-1} + q_{-2}} = \frac{a_0 \cdot 1}{1} = a_0.$$

Suppose it holds for all k.

$$[a_0, a_1, \cdots, a_{k+1}]$$
$$= [a_0, a_1, \cdots, a_{k-1}, a_k + \frac{1}{a_{k+1}}] = \frac{p_k'}{q_k'}$$

$$= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1}' + p_{k-2}'}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1}' + q_{k-2}'}$$

$$= \frac{(a_k a_{k+1} + 1) p_{k-1}' + a_{k+1} p_{k-2}'}{(a_k a_{k+1} + 1) q_{k-1}' + a_{k+1} q_{k-2}'}$$

$$= \frac{a_{k+1}(a_k p_{k-1}' + p_{k-2}') + p_{k-1}'}{a_{k+1}(a_k q_{k-1}' + q_{k-2}') + q_{k-1}'}$$

$$= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

$$= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}$$

$$= \frac{p_{k+1}}{q_{k+1}}$$

Real number x, compute integers $a_0, a_1, \cdots$, such that $a_0 = \lfloor x \rfloor$.

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

$x_1 = \frac{1}{x - a_0}$, then $a_1 = \lfloor x_1 \rfloor$,

$x_2 = \frac{1}{x_1 - a_1}$, then $a_2 = \lfloor x_2 \rfloor, \cdots$

Theorem $p_{k-1}q_k - q_{k-1}p_k = (-1)^k$.

Proof. (By PMI)

$$p_{-2}q_{-1} - q_{-2}p_{-1} = (-1)^{-1}.$$

Assume it holds for k.

$$p_k q_{k+1} - q_k p_{k+1}$$
$$= p_k(a_{k+1}q_k + q_{k-1}) - q_k(a_{k+1}p_k$$
$$+ p_{k-1}) = p_k q_{k-1} - q_k p_{k-1}$$
$$= -(q_k p_{k-1} - p_k q_{k-1}) = (-1)(-1)^k$$
$$= (-1)^{k+1}$$

Corollary $\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}$

Corollary $gcd(p_k, q_k) = 1$

If p_k and q_k had a nontrivial common divisor it would divide $p_k q_{k-1} - q_k p_{k-1}$, which is impossible.

Corollary $p_{k-2}q_k - q_{k-2}p_k = (-1)^{k-1}a_k$

Proof.

$$p_{k-1}q_k - q_{k-1}p_k = (-1)^k$$
$$a_k p_{k-1}q_k - a_k q_{k-1}p_k = (-1)^k a_k$$
$$(p_k - p_{k-2})q_k - (q_k - q_{k-2})p_k$$
$$= (-1)^k a_k$$
$$p_{k-2}q_k - q_{k-2}p_k = (-1)^{k+1}a_k$$

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1}a_k}{q_{k-2}q_k}$$

$$= \begin{cases} > 0 & k \text{ is odd} \\ < 0 & k \text{ is even} \end{cases}$$

$$\Rightarrow \frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} \cdots$$

$$\Rightarrow \frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} \cdots$$

Even terms increasing, bounded above by odd terms, odd terms decreasing, bounded below by even terms, so they both converge and get arbitrarily close.

So both even and odd sequences converge to the same real number x, namely,

$\frac{p_n}{q_n} \to x$ as $n \to \infty$.

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

$$\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} \to 0 \; as \; n \to \infty$$

$$\left|x - \frac{p_n}{q_n}\right| \leq \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}.$$

Theorem One of every 2 consecutive convergents satisfies

$$\left|x - \frac{p_k}{q_k}\right| \le \frac{1}{2q_k^2}$$

Proof.

$$\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} \le \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

$$\left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right|$$

$$\le \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

$$\Rightarrow \left|x - \frac{p_n}{q_n}\right| \le \frac{1}{2q_n^2} \ or \ \left|x - \frac{p_{n+1}}{q_{n+1}}\right| \le \frac{1}{2q_{n+1}^2}$$

Theorem One of every 3 consecutive convergents satisfies

$$\left|x-\frac{p_k}{q_k}\right| \leq \frac{1}{\sqrt{5}q_k^2}$$

Proof. Suppose it is not.

$\left|x-\frac{p_k}{q_k}\right| > \frac{1}{\sqrt{5}q_k^2}$ for $n, n+1, n+2$.

$$\left|x-\frac{p_n}{q_n}\right| + \left|x-\frac{p_{n+1}}{q_{n+1}}\right| = \left|\frac{p_n}{q_n}-\frac{p_{n+1}}{q_{n+1}}\right|$$

$$= \frac{1}{q_n q_{n+1}} > \frac{1}{\sqrt{5}q_n^2} + \frac{1}{\sqrt{5}q_{n+1}^2}$$

$$\Rightarrow \sqrt{5} > \frac{q_{n+1}}{q_n} + \frac{q_n}{q_{n+1}}$$

$$\Rightarrow \frac{q_{n+1}}{q_n} < \frac{\sqrt{5}+1}{2}$$

$f(x) = x + \frac{1}{x}$ is strictly increasing on $(1, \infty)$, so

$$\frac{q_n}{q_{n+1}} = \frac{1}{\frac{q_{n+1}}{q_n}} > \frac{\sqrt{5}-1}{2}$$

Likewise, $\frac{q_{n+2}}{q_{n+1}} < \frac{\sqrt{5}+1}{2}$, but

$$\frac{q_{n+2}}{q_{n+1}} = \frac{a_{n+2}q_{n+1} + q_n}{q_{n+1}} = a_{n+2} + \frac{q_n}{q_{n+1}}$$
$$\geq 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2}$$

Contradiction.

Why are continued fractions useful?

- Gives good approximations to real numbers
  - $\pi = [3,7,15,1,292,1,1,1,2,1,3,1,\cdots]$
  - $e = [2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,\cdots]$
- Useful in number theory for study of quadratic fields, diophantine equations
  - A real number x is a quadratic surd $\left(\frac{P+\sqrt{D}}{Q}\right)$ if and only if its continued fraction is periodic $[a_0; a_1, a_2, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+m}}]$.

Excise017

Please write $\sqrt{71}$ into the form of continued fraction.