卡特兰数

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平衡括号

n=1

()

n=2

()() (())

n=3

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(()()) ((()))

Notes: “(())()” is valid, but “())(()” is not.

n=4

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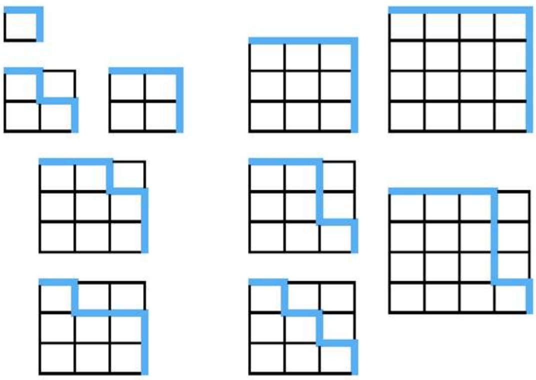
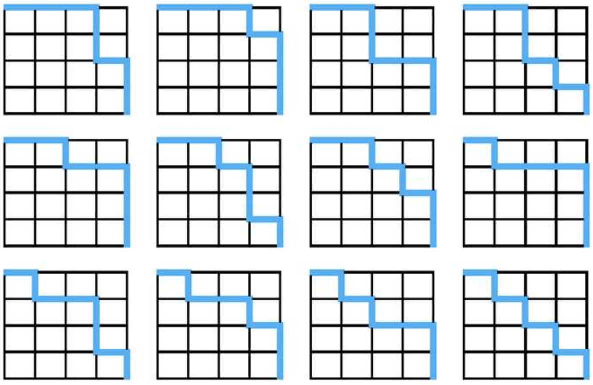
There is a sequence 1, 2, 5, 14, …

It is **Catalan numbers**, denoted by C_n.

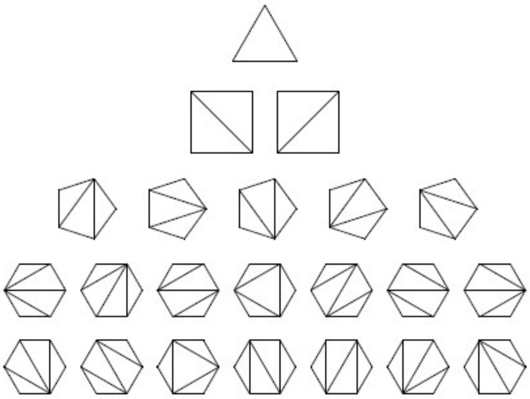
If We define the count for $n = 0$ to be 1,
then the sequence becomes 1, 1, 2, 5, 14, …



Monotonic lattice paths. A monotonic path which starts in the upper left corner, moves along the edges of a grid to the lower right corner, and do not pass across the diagonal.

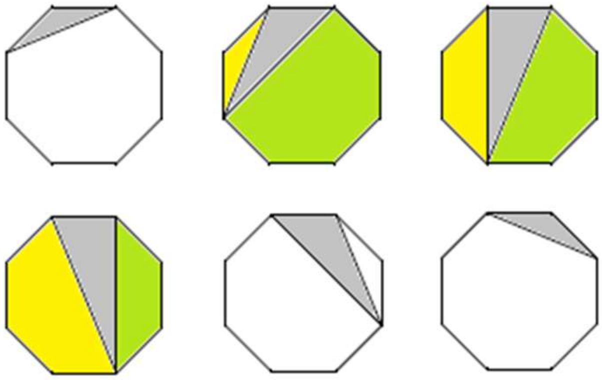


(0,0,0,2)(0,0,0,1)(0,0,2,2)(0,0,2,3)(0,0,1,1)(0,0,1,3)(0,0,1,2)(0,1,1,1)(
0,1,1,3) (0,1,1,2)(0,1,2,2)(0,1,2,3)



Polygon triangulation. (This is **Euler's polygon division problem**.)

递归数列



Counting **Polygon triangulation** of Octagon ($C_{8-2}=C_6$).
($C_6=C_0C_5+C_1C_4+C_2C_3+C_3C_2+C_4C_1+C_5C_0$)

It is observed that all of the combinatorial problems listed above satisfy **Segner's recurrence relation**

$$C_{n+1} = \sum_{i=0}^{n} C_i \; C_{n-i} \quad \text{for } n \geq 0,$$

together with $C_0 = 1$ and $C_1 = 1$.

生成函数

The **generating function** for the Catalan numbers is defined by

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

The two recurrence relations together can then be summarized in generating function form by the relation

$$c^2(x) = C_0C_0 + (C_1C_0 + C_0C_1)x$$
$$+(C_2C_0 + C_1C_1 + C_0C_2)x^2 + \cdots$$

$$c^2(x) = C_1 + C_2x + C_3x^2 + \cdots$$

$$c(x) = 1 + xc(x)^2$$

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n$$

$$= \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{2n-3}{2}\right)}{n!}(-4x)^n$$

$$= \sum_{n\geq 0} (-1)^{n-1} \frac{(2n-3)!!}{2^n n!}(-4x)^n$$

$$= -\sum_{n\geq 0} \frac{2^n (2n-3)!!}{n!} x^n$$

$$= -2\sum_{n\geq 0} \frac{2^{n-1}\prod_{k=1}^{n-1}(2k-1)}{n(n-1)!} x^n$$

$$= -2\sum_{n\geq 0} \frac{2^{n-1}(n-1)!\prod_{k=1}^{n-1}(2k-1)}{n(n-1)!^2} x^n$$

$$= -2\sum_{n\geq 0} \frac{\left(\prod_{k=1}^{n-1}(2k)\right)\left(\prod_{k=1}^{n-1}(2k-1)\right)}{n(n-1)!^2} x^n$$

$$= -2\sum_{n\geq 0} \frac{(2n-2)!}{n(n-1)!^2} x^n$$

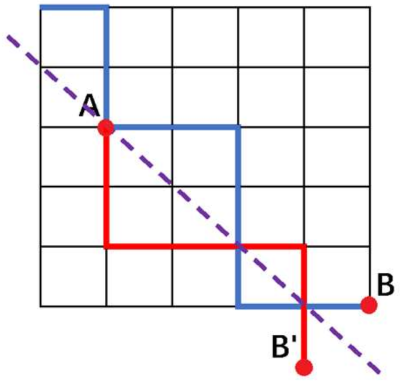
$$= -2\sum_{n\geq 0} \frac{1}{n}\binom{2n-2}{n-1} x^n,$$

$$
\begin{aligned}
c(x) &= \frac{1}{2x}\left(1 + 2\left(-\frac{1}{2} + \sum_{n\geq 1}\frac{1}{n}\binom{2(n-1)}{n-1}x^n\right)\right) \\
&= \sum_{n\geq 1}\frac{1}{n}\binom{2(n-1)}{n-1}x^{n-1} \\
&= \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n.
\end{aligned}
$$

通项公式

The formula of Catalan numbers

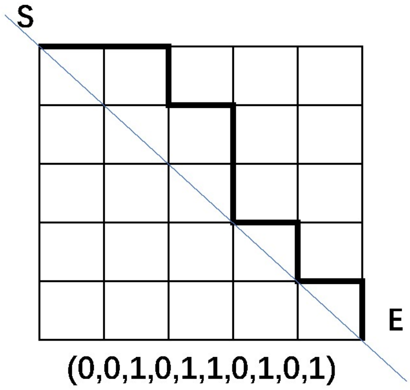
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



Counting **Monotonic lattice paths** using **Path Reflection**

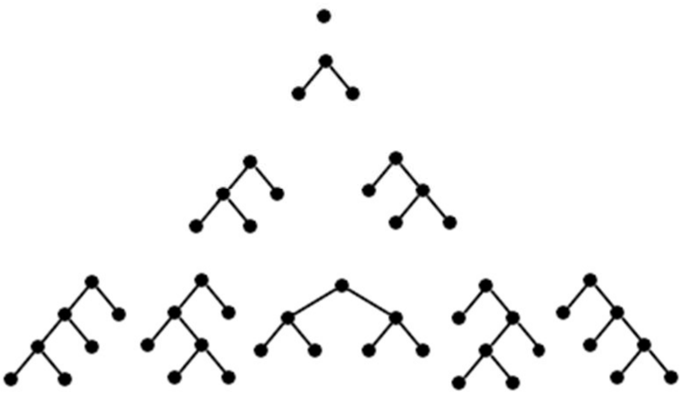
The expression for C_n is

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$



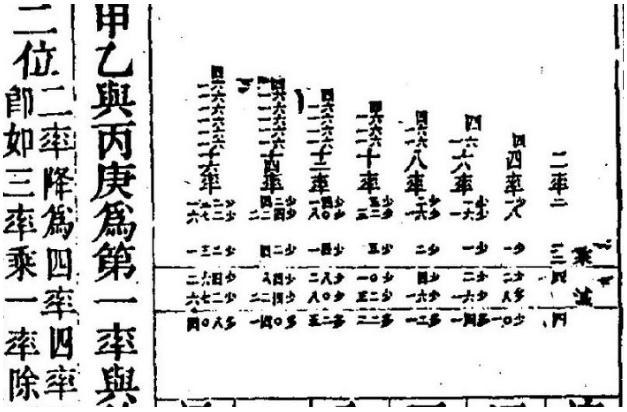
(0,0,1,0,1,1,0,1,0,1)

One can count the **balanced parentheses** by “0” -> “(”, “1” -> “)”



Rooted binary trees with n internal nodes

明安图



Ming An tu (1692-1763) first established and used what was later to be known as Catalan numbers in 1730.

卡特兰数列表

```
n<-c(1:20)
4^n*gamma(n+1/2)/(sqrt(pi)*gamma(2+n)
)
```

```
 [1]          1          2          5
14
 [5]         42        132        429
1430
 [9]       4862      16796      58786
208012
[13]     742900    2674440    9694845
35357670
[17]  129644790  477638700 1767263190
6564120420
```