生成函数

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递推关系

We have deposited 1000 yuan into a bank account that pays 5% interest at the end of each year.

At the beginning of each year, we deposit another 500 yuan into this account.

How much money will be in this account after 20 years?

$$a_0 = 1000$$
$$a_{n+1} = 1.05 \cdot a_n + 500$$

$$a_{20} = ?$$

Excise C015

$$a_0 = 1000$$

$$a_{n+1} = 1.05 \cdot a_n + 500$$

$$a_{20} = ?$$

Derangement

Suppose that n persons are numbered $1,2,\cdots,n$. Let there be n hats also numbered $1,2,\cdots,n$. We have to find the number of ways in which no one gets the hat having same number as his/her number.

Let A_i be the set of all permutations of [n] in which the element i is in the ith position, in other words, in which the element i is fixed.

For example, $23541 \in A_4$.

The answer is connected with $\#(\bigcup_{i=1}^{n} A_i)$.

$$\#A_i = (n-1)!$$

The set $A_i \cap A_j$ consists of permutations in which elements i and j are fixed, and the remaining $n-2$ entries can be permuted freely, in $(n-2)!$ ways.

$$\#(A_i \cap A_j) = (n-2)!$$

So, the solution is

$$n! - \left[\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \cdots\right]$$

or

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!$$

or

$$d_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

Recursion Suppose we want to determine the number of derangements of the n integers $1,2,\cdots,n$.

Let us focus on k and move it into the first position.

We thus have started a derangement, for 1 is not in its natural position. Where could 1 be placed?

There are two cases we could consider: either 1 is in position k or 1 is not in position k.

If 1 is in position k,

$$\begin{matrix} 1 & 2 & 3 & \cdots & k-1 & k & k+1 & \cdots & n \\ k & ? & ? & \cdots & ? & 1 & ? & \cdots & ? \end{matrix}$$

There are $n-2$ integers yet to derange.

This can be done in d_{n-2} ways.

If 1 is not in position k,

$$\begin{matrix} 1 & 2 & 3 & \cdots & k-1 & k & k+1 & \cdots & n \\ k & ? & ? & \cdots & ? & ? & ? & \cdots & ? \end{matrix}$$

There are now $n-1$ integers to derange.

This can be done in d_{n-1} ways.

Putting this together, we have $d_{n-1} + d_{n-2}$ possible derangements when k is in the first position.

There are $(n-1)$ different ways to select the k element.

So,

$$d_n = (n-1)[d_{n-1} + d_{n-2}]$$

$d_1 = 0$ and $d_2 = 1$

普通生成函数

A **generating function** is a formal **power series** in one indeterminate, whose coefficients encode information about a sequence of numbers a_n that is indexed by the natural numbers.

The **ordinary generating function** of a sequence a_n is

$$G(a_n, x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example $a_0 = 50$, $a_{n+1} = 4a_n - 100$.
Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} 4\, a_n x^{n+1} - \sum_{n=0}^{\infty} 100\, x^{n+1}$$

$$G(x) - a_0 = 4xG(x) - \frac{100x}{1-x}.$$

$$G(x) = \frac{a_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}$$

$$\frac{a_0}{1-4x} = 50 \sum_{n=0}^{\infty} (4x)^n = 50 \sum_{n=0}^{\infty} 4^n x^n$$

$$\frac{100x}{(1-x)(1-4x)} = -\frac{100/3}{1-x} + \frac{100/3}{1-4x} = \frac{100}{3}\left(\sum_{n=0}^{\infty} 4^n x^n - \sum_{n=0}^{\infty} x^n\right)$$

$$G(x) = 50\sum_{n=0}^{\infty} 4^n x^n - \frac{100}{3}\left(\sum_{n=0}^{\infty} 4^n x^n - \sum_{n=0}^{\infty} x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(50\cdot 4^n - 100\cdot\frac{4^n-1}{3}\right)x^n$$

Since $G(x) = \sum_{n=0}^{\infty} a_n x^n$,

$$a_n = 50\cdot 4^n - 100\cdot\frac{4^n-1}{3}.$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\frac{A}{1-x} + \frac{B}{1-4x} = \frac{100x}{(1-x)(1-4x)}$$

$$A(1-4x) + B(1-x) = 100x$$
$$(-B-4A)x + A + B = 100x$$

$$\begin{cases} -B-4A = 100 \\ A+B = 0 \end{cases}$$

Excise C016

Please compute a_{20} using generating function.

$$a_0 = 1000$$

$$a_{n+1} = 1.05 \cdot a_n + 500$$

斐波那契数列

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

A skew Yanghui's Triangle

The sequence Fn of **Fibonacci numbers** is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with initial values:

$$F_1 = 1,\ F_2 = 1.$$

The generating function of the **Fibonacci sequence** is the power series

$$G(x) = \sum_{k=0}^{\infty} F_k\, x^k.$$

Next

$$= F_0 + F_1 x + \sum_{k=2}^{\infty} (F_{k-1} + F_{k-2})\, x^k$$

$$= x + \sum_{k=2}^{\infty} F_{k-1}\, x^k + \sum_{k=2}^{\infty} F_{k-2}\, x^k$$

$$= x + x\sum_{k=0}^{\infty} F_k\, x^k + x^2 \sum_{k=0}^{\infty} F_k\, x^k$$

$$= x + xG(x) + x^2 G(x).$$

Solving the equation

$$G(x) = x + xG(x) + x^2 G(x)$$

Then

$$G(x) = \frac{x}{1 - x - x^2}$$

Excise C017

Try to obtain the **explicit** formula for the Fibonacci number F_n.

R program for listing the Fibonacci numbers.

```
n <- 20
myFib <- numeric(n)
myFib[1] <- 1
myFib[2] <- 1
for (i in 3:n) {
   myFib[i] <- myFib[i-1]+myFib[i-2]
}
myFib
 [1]    1    1    2    3    5    8
13   21   34   55
[11]   89  144  233  377  610  987
1597 2584 4181 6765
```

Theorem The sum of the first n Fibonacci numbers can be expressed as

$$F_1 + F_2 + \cdots + F_{n-1} + F_n = F_{n+2} - 1.$$

Proof.

$$F_1 = F_3 - F_2$$
$$F_2 = F_4 - F_3$$
$$F_3 = F_5 - F_4$$
$$\cdots$$
$$F_n = F_{n+2} - F_{n+1}$$

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - F_2 = F_{n+2} - 1$$

Theorem The sum of the odd terms of the Fibonacci sequence

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

Excise C018

Can you prove it?

Theorem The sum of the even terms of the Fibonacci sequence

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Proof.

$$F_1 + F_2 + \cdots + F_{2n} = F_{2n+2} - 1$$
$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$$

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+2} - 1 - F_{2n}$$
$$= F_{2n+1} - 1$$

Theorem The sum of the Fibonacci numbers with alternating signs

$$F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1.$$

Excise C019

Can you prove it?

Theorem The sum of the squares of the first n Fibonacci numbers

$$F_1^2 + F_2^2 + \cdots + F_{n-1}^2 + F_n^2 = F_n F_{n+1}$$

Proof.

$$F_k^2 = F_k(F_{k+1} - F_{k-1}) = F_k F_{k+1} - F_k F_{k-1}$$

$$F_1^2 = F_1 F_2$$
$$F_2^2 = F_2 F_3 - F_1 F_2$$
$$F_3^2 = F_3 F_4 - F_2 F_3$$
$$\cdots$$
$$F_n^2 = F_n F_{n+1} - F_{n-1} F_n$$

Johannes Kepler observed that the ratio of consecutive Fibonacci numbers converges.

The limit approaches the golden ratio φ.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

Proof.

Using the Fibonacci rule:

$$F_n = F_{n-1} + F_{n-2}.$$

$$\varphi^2 = \varphi + 1,$$

$$\varphi = \frac{1 \pm \sqrt{5}}{2} \approx 1.618 \, or \, -0.618.$$

指数生成函数

The **exponential generating function** of a sequence a_n is

$$\mathrm{E}(a_n, x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Example Let $a_0 = 1$, and let

$$a_{n+1} = (n+1)(a_n - n + 1),$$

if $n > 0$. Let is try to find a closed formula for a_n.

The exponential generating function is

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!}$$

$$E(x) - 1 = xE(x) - x^2e^x + xe^x$$

$$E(x) = \frac{1}{1-x} + xe^x = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

$$E(x) = \sum_{n=0}^{\infty} n!\frac{x^n}{n!} + \sum_{n=0}^{\infty} (n+1)\frac{x^{n+1}}{(n+1)!}.$$

$$E(x) = \sum_{n=0}^{\infty} n!\frac{x^n}{n!} + \sum_{n=0}^{\infty} n\frac{x^n}{n!}.$$

Since $E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$,

$$a_n = n! + n.$$

Excise C020

Let $a_0 = 0$. If $n > 0$, then

$$a_{n+1} = 2(n+1)a_n + (n+1)!$$

Try to find a closed formula for a_n.

Derangement

Start with the recurrence

$$d_{n+1} = nd_n + nd_{n-1}.$$

Define $d_{-1} = 0$ which is consistent with the recurrence.

Multiply by $\frac{x^n}{n!}$ and sum over $n \geq 0$,

$$\sum_{n\geq 0} d_{n+1} \frac{x^n}{n!}$$
$$= \sum_{n\geq 0} n\, d_n \frac{x^n}{n!} + \sum_{n\geq 0} n\, d_{n-1} \frac{x^n}{n!}.$$

Let

$$E(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!}$$

be the exponential generating function.

$$E'(x) = \sum_{n\geq 0} n\, d_n \frac{x^{n-1}}{n!} = \sum_{n\geq 1} d_n \frac{x^{n-1}}{(n-1)!}$$
$$= \sum_{n\geq 0} d_{n+1} \frac{x^n}{n!},$$
$$xE'(x) = x \sum_{n\geq 0} n\, d_n \frac{x^{n-1}}{n!} = \sum_{n\geq 0} n\, d_n \frac{x^n}{n!},$$
$$xE(x) = \sum_{n\geq 0} d_n \frac{x^{n+1}}{n!}$$

$$= \sum_{n\geq 0} (n+1) d_n \frac{x^{n+1}}{(n+1)!} = \sum_{n\geq 1} n\, d_{n-1} \frac{x^n}{n!}$$
$$= \sum_{n\geq 0} n\, d_{n-1} \frac{x^n}{n!}$$

So,

$$E'(x) = xE'(x) + xE(x),$$
$$\frac{E'(x)}{E(x)} = \frac{x}{1-x} = -1 + \frac{1}{1-x},$$
$$E(x) = \frac{e^{-x}}{1-x}.$$

$$E(x) = (1 + x + x^2 + \cdots + x^n + \cdots) \cdot \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right] x^n$$

$$d_n = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right] \cdot n!$$