排列与组合

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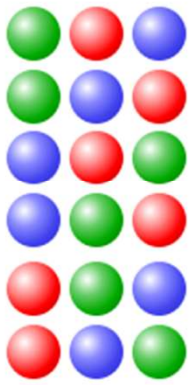
排列与阶乘

The arrangement of different objects into a linear order using each object exactly once is called a **permutation** of these objects.

- There are n choices for the first place.
- We have $n-1$ choices for the second place.
- We have $n-2$ choices for the third place.
- ……
- There is only one choice for the last place.



Permutation of three balls with different colors.

The number $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ of all permutations of n objects is called n **factorial**, and is denoted by $n!$.

Define

$$0! = 1.$$

Theorem The number of all permutations of an n-element set is $n!$.

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Excise C006

A gardener has 3 blue flowers, 4 red flowers and 5 white flowers to plant in a row. In how many different ways can she do that?

Let $n, k, a_1, a_2, \cdots, a_k$ be non-negative integers satisfying $a_1 + a_2 + \cdots + a_k = n$. Consider a **multiset** of n objects, in which a_i objects are of **type** i, for all $i \in [k]$. Then the number of ways to linearly order these objects is

$$\frac{n!}{a_1!\, a_2! \cdots a_k!}$$

字符串

Next, we construct strings, or words, from a finite set of symbols, which we call a **finite alphabet**.

We will not require that each symbol occur a specific number of times.

Theorem The number of k-digit strings one can form over an n-element **alphabet** is n^k.

Excise C007

How many 6-digit positive integers are there in decimal system?

For a pairing between X and Y to be a bijection, four properties must hold:

- each element of X must be paired with at least one element of Y,
- no element of X may be paired with more than one element of Y,
- each element of Y must be paired with at least one element of X, and
- no element of Y may be paired with more than one element of X.

Satisfying properties (1) and (2) means that a **bijection** is a **function** with **domain** X.

Functions which satisfy property (3) are said to be "onto Y" and are called **surjections**.

Functions which satisfy property (4) are said to be "one-to-one functions" and are called **injections**.

Example The number of all subsets of an n-element set is 2^n.

Let B be any subset of [n], and let f(B) be the string whose ith digit is 1 if and only if $i \in B$ and 0 otherwise.

The latter set has 2^n elements.

Suppose n=9.

$$B = \{1,3,5,6\} \Leftrightarrow f(B) = '101011000'$$
$$B = \{2,5,7,9\} \Leftrightarrow f(B) = '010010101'$$

Theorem Let n and k be positive integers satisfying $n > k$. Then the number of k-digit strings over an n-element alphabet in which no letter is used more than once is

$$P(n,k) = n(n-1)\cdots(n-k+1)$$
$$= \frac{n!}{(n-k)!}$$

These objects are also known as **k-permutations of n**, or partial permutations.

Example There are 10 boxes in different colors. In how many different ways can each of Jack, Rose, Smith, Hilary, and Jessica choose one box from them?

$$P(10,5) = \frac{10!}{5!}$$

组合

A **k-element combination** of an n-set S is a k element subset of S, the elements of which are not ordered.

For example, choose 5 cards from a suit to form a particular hand.

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!\,k!}$$

Theorem For all non-negative integers $k < n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

hold. And

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Excise C008

A worker has to work for 6 days in May.

However, he is not allowed to work two consecutive days.

In how many different ways can he choose the 6 days he will work?

Excise C009

Assume that we play a lottery game where 5 numbers are drawn out of [100], but the numbers drawn are put back.

To win the jackpot, one must have played the same multiset of numbers as the one drawn (regardless of the order).

How many lottery tickets do we have to buy to make sure that we win the jackpot?

(5,93,66,34,66 is the same as 66,66,34,93,5.)

Theorem The number of **k-element multisets** whose elements all belong to [n] is

$$\binom{n+k-1}{k}$$

The **k-element combination** is also known as **binomial coefficients**.

二项式系数

Theorem For all non-negative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

write $(x+y)^n$ as a product

$$(x+y)(x+y)(x+y)\cdots(x+y).$$

There will be one term in the expansion for each choice of either x or y. The **binomial coefficient** $\binom{n}{k}$ can be interpreted as the number of ways to choose k elements from an n-element set.

Example

The coefficient of xy^2 in

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= xxx + xxy + xyx + \boxed{xyy} + yxx + \boxed{yxy} + \boxed{yyx} + yyy$$
$$= x^3 + 3x^2y + \boxed{3xy^2} + y^3.$$

equals $\binom{3}{2} = 3$.

Theorem

Let n > 0,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof.

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k.$$

Theorem

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof.

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k.$$

Recursive formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for all integers n, k: $1 \leq k \leq n-1$, with initial values

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{for all integers } n \geq 0.$$

Excise C010

Try to prove the recursive formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Combinatorial proof of **Recursive formula**.

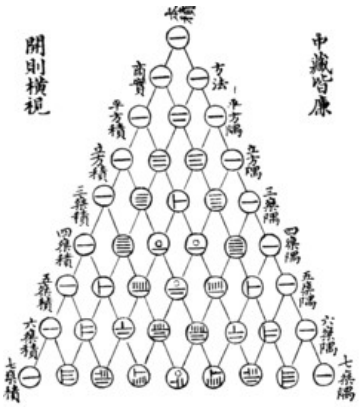
The left-hand side is the number of k-element subsets of $[n]$.

Such a subset S either contains n, or not.

- If it does, then the rest of S is a $k-1$-element subset of $[n-1]$, and these are counted by the first part of the right-hand side.
- If it does not, then S is a k-element subset of $[n-1]$, and these are enumerated by the second part of the right-hand side.

Recursive formula also gives rise to Yanghui's triangle:

$$\{1\}$$
$$\{1,1\}$$
$$\{1,2,1\}$$
$$\{1,3,3,1\}$$
$$\{1,4,6,4,1\}$$
$$\{1,5,10,10,5,1\}$$
$$\{1,6,15,20,15,6,1\}$$
$$\{1,7,21,35,35,21,7,1\}$$



The original Yanghui's Triangle.

Excise C011

Try to prove using mathematical induction:

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

Hockey-stick identity

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

Combinatorial proof.

The right-hand side counts the number of $k + 1$-element subsets of $[n + 1]$.

- That is, there are $\binom{n}{k}$ subsets of $[n + 1]$ that have $k + 1$ elements whose **largest element** is $n + 1$.
- There are $\binom{k+i}{k}$ subsets of $[n + 1]$ that have $k + 1$ elements whose **largest element** is $k + i + 1$.

Excise C012

Try to prove: for all non-negative integers n,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Theorem $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Combinatorial proof.

Choose a committee among n people, then to choose a president from the committee.

- Left-hand side: we first choose a k-member committee in $\binom{n}{k}$ ways, then we choose its president in k ways.
- Right-hand side: we first choose the president in n ways, then we choose a subset of the remaining $n-1$-member set of people for the role of non-president committee members in 2^{n-1} ways.

Chu–Vandermonde identity For any complex-values m and n and any non-negative integer k,

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}$$

Combinatorial proof.

- Right-hand side: all k-element subsets of $[n]$.
- Left-hand side: We can first choose j elements from $[m]$ in $\binom{m}{j}$ ways, then choose the remaining $k-j$ elements from the set $\{m+1, m+2, \cdots, n\}$ in $\binom{n-m}{k-j}$ ways.

Theorem For all non-negative integers k and n, such that $k \leq \frac{n-1}{2}$, the inequality

$$\binom{n}{k} \leq \binom{n}{k+1}$$

holds. Furthermore, equality holds if and only if $n = 2k+1$.

$$\frac{n!\,(\boxed{k+1})}{(k+1)!\,(n-k)!} \qquad \frac{n!\,(\boxed{n-k})}{(k+1)!\,(n-k)!}$$

Corollary For all non-negative integers k and n, such that $k \geq \frac{n-1}{2}$, the inequality

$$\binom{n}{k} \geq \binom{n}{k+1}$$

holds. Furthermore, equality holds if and only if $n = 2k + 1$.

多项式系数

Multinomial coefficients

The third power of the trinomial $x + y + z$ is given by

$$(x + y + z)^3$$

$$= x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3y^2x + 3y^2z + 3z^2x + 3z^2y + 6xyz.$$

Binomial coefficients can be generalized to multinomial coefficients defined to be:

$$\binom{n}{a_1, a_2, \cdots, a_k} = \frac{n!}{a_1!\, a_2! \cdots a_k!}$$

where $\sum_{i=1}^{k} a_i = n$.

$$(x_1 + x_2 + \cdots + x_k)^n$$

$$= \sum_{a_1+a_2+\cdots+a_k=n} \binom{n}{a_1, a_2, \ldots, a_k} \prod_{1\le i\le k} x_i^{a_i},$$

The case $k = 2$ gives binomial coefficients:

$$\binom{n}{a_1, a_2} = \binom{n}{a_1, n-a_1} = \binom{n}{a_1} = \binom{n}{a_2}.$$

Theorem For all non-negative integers n and $a_1, a_2, \cdots, a_k$ such that $\sum_{i=1}^{k} a_i = n$.

$$\binom{n}{a_1, a_2, \ldots, a_k} =$$

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{k-1}}{a_k}$$

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{k-1}}{a_k} =$$

$$\frac{n!}{\boxed{(n-a_1)!}\,a_1!}\,\frac{\boxed{(n-a_1)!}}{\boxed{(n-a_1-a_2)!}\,a_2!}\,\frac{\boxed{(n-a_1-a_2)!}}{(n-a_1-a_2-a_3)!\,a_3!}\cdots$$